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LETTER TO THE EDITOR

Wavelet multiresolutions for the Fibonacci chain

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Abstract. Most of the discrete wavelet families are periodic dyadic: they live on the multiresolution sequence $\{\mathbb{Z}/2^j\}_{j\in\mathbb{Z}}$ of lattice backgrounds at different scales. We present here a construction of wavelet family based on a sequence of aperiodic discretizations of $\mathbb{R}$. At a given scale, this discretization is the point set of nodes of the Fibonacci chain, a well known stone-inflation cut and project model for one-dimensional quasicrystals. Corresponding multiresolution analysis and the elementary example of the Haar system are presented.

1. Introduction

Dyadic wavelets are the most commonly studied and used in discrete wavelet analysis. Excellent textbooks devoted to the subject exist on the market. We refer in particular to [1], for the notation and some of the standard results. The dyadic wavelet analysis is based on the following increasing sequence of periodic discretizations of $\mathbb{R}$: $\cdots \subset \mathbb{Z}/2^{j-1} \subset \mathbb{Z}/2^j \subset \mathbb{Z}/2^{j+1} \subset \cdots$. We present here an example of discrete wavelet analysis where the multiresolution involves an aperiodic irrational discretization of the set of real numbers. The scaling factor is the square of the golden ratio, $\tau^2 = (3+\sqrt{5})/2$, and at a given scale, say the zeroth scale, the discretization set is the set of nodes of the Fibonacci chain. The latter can be obtained from the square lattice $\mathbb{Z}^2$ by cut and project, a method familiar to quasicrystallographers. It is equivalently obtained from substitution rules. This work is in a sense a continuation of previous tentative attempts at construction of wavelets adapted to simple quasicrystalline models [2]. More precisely, in the first item of [2] a construction was presented of the Haar system adapted to the sequence $\{\mathbb{Z}_\tau/\tau^j\}_{j\in\mathbb{Z}}$ of aperiodic discretizations of the real line with scaled versions of the set of 'tau-integers'. In relation to that, the novelty of the present approach has two aspects. Firstly, the construction is based on the Fibonacci chain. This is a stone-inflation tiling of the real line with no privileged origin (contrary to the symmetric $\mathbb{Z}_\tau = -\mathbb{Z}_\tau$). Secondly, we properly define a multiresolution analysis with *two* scaling functions (one occasionally says *father wavelets*), one per ('long' or 'short') type of tile, and this prevents inconsistencies in setting up the related scaling equations (see equation (3.5) below). In the meantime, our construction rests upon affine-linear invariance properties of the Fibonacci chain, and this provides a nice illustration of the algebraic approach initiated by Moody and Patera in [3] (see (2.4)–(2.7) below).

The construction of the Fibonacci chain by substitution will be recalled in the next section, together with the main algebraic properties of this toy model of quasicrystals. In section 3, we shall define what we mean by multiresolution analysis based on the infinite sequence of scaled

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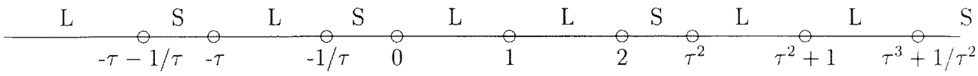

Figure 1.

versions of the Fibonacci chain, and we shall describe the scaling conditions from which it becomes possible to build up the corresponding wavelet family. The last section is devoted to the Haar system which, although elementary, provides us with a nice illustration of the general construction.

2. The sequence of Fibonacci discretizations

The 'zero' scale $\Lambda \equiv \Lambda_0$ of what we mean by Fibonacci discretization is the set of left-hand ends of the two-letter Fibonaccci tiling of the line. This tiling originates from the substitution

$$\varsigma : \begin{cases} L \to LLS \\ S \to LS \end{cases} \tag{2.1}$$

for which the substitution matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

has eigenvalues $\tau^2 = (3 + \sqrt{5})/2$ and $1/\tau^2 = (3 - \sqrt{5})/2$. Therefore, the Pisot algebraic integer τ^2 is the scaling factor in the self-similar properties of ς. Let us associate to the letters L and S the tiles of length 1 and $1/\tau$ respectively. Starting from the origin with L on the right and with S on the left, we apply the substitution ς^∞ to the former on its right and to the latter on its left. We thus get the point set Λ as the set of left-hand ends of this tiling of the line. The first pieces of this tiling around the origin are shown in figure 1.

In algebraic terms, the set Λ, as a subset of the extension ring $\mathbb{Z}[\tau]$, is the set of all numbers $m + n\tau$, $m, n \in \mathbb{Z}$, such that their respective Galois conjugates $m - n(1/\tau)$ lie within the semi-open interval (i.e. the cut and project window) $[0, \tau^2)$. We shall here adopt the notations of Moody and Patera [3]: $\Lambda = \Sigma^{[0,\tau^2)}$. Note the stone-inflation self-similarity

$$\tau^2\Lambda = \Sigma^{[0,1)} \subset \Lambda \tag{2.2}$$

a crucial property for our wavelet purposes. Indeed, the 'τ^2-adic' multiresolution is based on the increasing sequence of discretizations of $\mathbb{R}$:

$$\cdots \subset \Lambda/\tau^{2(j-1)} \subset \Lambda/\tau^{2j} \subset \Lambda/\tau^{2(j+1)} \subset \cdots. \tag{2.3}$$

Actually, equation (2.2) is a particular case of the general affine-linear invariance property enjoyed by the point set Λ:

$$\tau^2\Lambda + \Sigma^{[0,\tau]} = \Lambda. \tag{2.4}$$

We now consider the partition of Λ into two subsets, $\Lambda = \Lambda_L \cup \Lambda_S$, where Λ_L (resp. Λ_S) is the set of left-hand ends of large (resp. short) tiles. Explicitly

$$\Lambda_L = \Sigma^{[0,\tau)} \qquad \Lambda_S = \Sigma^{[\tau,\tau^2)}. \tag{2.5}$$

Similarly to (2.2), these subsets have the following respective affine-linear invariances:

$$\tau^2\Lambda_L + \Sigma^{[0,1]} = \Lambda_L \qquad \tau^2\Lambda_L + \Sigma^{[\tau,2)} = \Lambda_S \tag{2.6}$$

$$\tau^2\Lambda_S + \Sigma^{[-1/\tau,1/\tau]} = \Lambda_L \qquad \tau^2\Lambda_S + \Sigma^{[1,\tau]} = \Lambda_S. \tag{2.7}$$

Table 1.

$\mu\backslash\nu$	L	S
L	$\Sigma^{[0,1]}$	$\Sigma^{[\tau,2]}$
S	$\Sigma^{[0,1+1/\tau^3]}$	$\Sigma^{[\tau,\tau+1/\tau]}$

3. The Fibonacci multiresolution

The multiresolution analysis of $L^2(\mathbb{R})$ which is based on the above Fibonacci discretization of $\mathbb{R}$ is the increasing sequence of closed subspaces

$$\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots \tag{3.1}$$

obeying the following set of requirements.

- *Density*

$$\cap_j V_j = \{0\} \qquad \overline{\cup_j V_j} = L^2(\mathbb{R}). \tag{3.2}$$

- *Scaling connection*

$$f(x) \in V_j \iff f(\tau^2 x) \in V_{j+1}. \tag{3.3}$$

- *Scaling functions*
  There exist two functions $\phi_L, \phi_S \in V_0$, the so-called *father wavelets*, such that the set

$$\{\phi_L(x-l), \phi_S(x-s)\}_{l\in\Lambda_L, s\in\Lambda_S} \tag{3.4}$$

  is an orthonormal basis of V_0 (or, at least, a Riesz basis).

The scaling equation resulting from the embedding $V_0 \subset V_1$ reads

$$\phi_\mu(x+\delta_{\mu S}/\tau) = \tau^2 \sum_{l\in T_L^\mu} c_{\mu l}\phi_L(\tau^2 x - l) + \tau^2 \sum_{s\in T_S^\mu} c_{\mu s}\phi_S(\tau^2 x - s) \qquad \mu \in \{L, S\}. \tag{3.5}$$

The presence of the translation parameter $\delta_{\mu S}/\tau$ in the above is due to the fact that 0 is element of Λ_L whereas $-1/\tau$ is element of Λ_S. Related to this remark, the admissible translation sets T_ν^μ appearing here are defined consistently to $\phi_\mu(x-\omega_\mu) \in V_0 \subset V_1$, $\omega_\mu \in \Lambda_\mu$, $\mu \in \{L, S\}$,

$$T_\nu^\mu = \{\omega_\nu | \omega_\nu + \delta_{\mu S}\tau + \tau^2\Lambda_\mu \subset \Lambda_\nu\}. \tag{3.6}$$

They are given in table 1.

Equation (3.5) can be the departure point for 'bare-hand' constructions of father wavelets in the spirit of chapter 4 in [1]. Indeed, if we restrict the values of x to the sets $-\Lambda_L$ and $-\Lambda_S$ respectively, we obtain a closed linear system precisely because of the nature of the T_ν^μ coupled to the affine-linear invariances (2.6) and (2.7):

$$\phi_\mu(-\omega_\mu) = \tau^2 \sum_{l\in T_L^\mu} c_{\mu l}\phi_L(-\tau^2\omega_\mu - l = -l') + \tau^2 \sum_{s\in T_S^\mu} c_{\mu s}\phi_S(-\tau^2\omega_\mu - s = -s')$$
$$\mu \in \{L, S\}. \tag{3.7}$$

The Fourier transform of equation (3.5) can be given a matrix form involving the vector function $\hat\Phi^{\mathrm{T}}(\xi) = (\hat\phi_L(\xi), \hat\phi_S(\xi))$ in reciprocal space:

$$\hat\Phi(\xi) = M_0\left(\frac{\xi}{\tau^2}\right)\hat\Phi\left(\frac{\xi}{\tau^2}\right) = \prod_{j\geqslant 1} M_0\left(\frac{\xi}{\tau^{2j}}\right)\hat\Phi(0). \tag{3.8}$$

The matrix $M_0(\xi)$ has almost-periodic entries,

$$m_{0\mu\nu}(\xi) = \sum_{\omega_\nu\in T_\nu^\mu} c_{\mu\omega_\nu} \mathrm{e}^{-2\pi\mathrm{i}(\omega_\nu+\delta_{\mu S}\tau)\xi}. \tag{3.9}$$

Equation (3.8), besides uniform convergence requirements, implies that $M_0(0)$ has eigenvalue 1 with eigenvector $\hat{\Phi}(0)$. The orthonormality and the completeness of the set (3.4) is reformulated in the reciprocal Fourier space as follows:

$$\int_{-\infty}^{\infty} \mathrm{d}\xi \begin{pmatrix} \mathrm{e}^{-2\pi \mathrm{i} l\xi} & 0 \\ 0 & \mathrm{e}^{-2\pi \mathrm{i} s\xi} \end{pmatrix} \hat{\Phi}(\xi)\hat{\Phi}(\xi)^{\dagger} \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} l'\xi} & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i} s'\xi} \end{pmatrix} = \begin{pmatrix} \delta_{ll'} & 0 \\ 0 & \delta_{ss'} \end{pmatrix}. \tag{3.10}$$

The meaning of (3.10) is that the Fourier transforms of the entries of the Hermitian matrix $\hat{\Phi}(\xi)\hat{\Phi}(\xi)^{\dagger}$, namely $|\hat{\phi}_L(\xi)|^2$, $\bar{\hat{\phi}}_L(\xi)\hat{\phi}_S(\xi)$, $|\hat{\phi}_S(\xi)|^2$, vanish respectively on the point sets:

$$\begin{aligned} &(\Lambda_L - \Lambda_L) \setminus \{0\} = \Sigma^{(-\tau,\tau)} \setminus \{0\} \qquad (\Lambda_S - \Lambda_L) = \Sigma^{(0,\tau^2)} \\ &(\Lambda_S - \Lambda_S) \setminus \{0\} = \Sigma^{(-1,1)} \setminus \{0\}. \end{aligned} \tag{3.11}$$

The wavelets ψ_L, ψ_S, occasionally named *mother wavelets*, are such that all their respective translates in $\{\psi_L(x-l), \psi_S(x-s)\}_{l\in\Lambda_L, s\in\Lambda_S}$ form an orthonormal basis of W_0 in $V_1 = V_0 \oplus_{\perp} W_0$. Their determination can be performed via the reciprocal space formulation involving the vector $\hat{\Psi}^{\mathrm{T}}(\xi) = (\hat{\psi}_L(\xi), \hat{\psi}_S(\xi))$.

$$\hat{\Psi}(\xi) = M_1\left(\frac{\xi}{\tau^2}\right)\hat{\Phi}\left(\frac{\xi}{\tau^2}\right). \tag{3.12}$$

The matrix $M_1(\xi)$ has almost-periodic entries too. The questions to address now are the following:

- What are the operational conditions to be imposed on $M_0(\xi)$ and $M_1(\xi)$ in view of solving this problem?
- Related to this, to what extent can we make use of the concept of Meyer 'ϵ-dual' sets T_{ϵ}^* [4] corresponding to the set T of frequencies of various almost-periodic functions involved in the above formulation?

Once these points are clarified, we shall be able to assert that the set $\{\tau^j\psi_L(\tau^{2j}x - l), \tau^j\psi_S(\tau^{2j}x - s)\}_{l\in\Lambda_L, s\in\Lambda_S, j\in\mathbb{Z}}$ is the expected orthonormal wavelet basis 'living' at the corresponding scale on the discretization sequence (2.3).

4. The Fibonacci Haar system

In the absence of rigorous answers to the above questions, let us illustrate the above material with an elementary example, namely the Haar system associated to the Fibonacci chain Λ. The scaling functions are just the normalized characteristic functions of large tile and short tile respectively:

$$\phi_L(x) = \mathbb{1}_{[0,1]}(x) \qquad \phi_S(x) = \tau^{1/2}\mathbb{1}_{[0,1/\tau]}(x). \tag{4.1}$$

The scaling equations read in the present case:

$$\phi_L(x) = \phi_L(\tau^2 x) + \phi_L(\tau^2 x - 1) + \tau^{-1/2}\phi_S(\tau^2 x - 2) \tag{4.2}$$

$$\phi_S(x) = \tau^{1/2}\phi_L(\tau^2 x) + \phi_S(\tau^2 x - 1). \tag{4.3}$$

Their Fourier transforms are given in the matrix form:

$$\hat{\Phi}(\xi) = \begin{pmatrix} \tau^{-2}(1+\mathrm{e}^{-2\pi\mathrm{i}\xi/\tau^2}) & \tau^{-5/2}\mathrm{e}^{-4\pi\mathrm{i}\xi/\tau^2} \\ \tau^{-3/2} & \tau^{-2}\mathrm{e}^{-2\pi\mathrm{i}\xi/\tau^2} \end{pmatrix} \hat{\Phi}\left(\frac{\xi}{\tau^2}\right) \equiv M_0\left(\frac{\xi}{\tau^2}\right)\hat{\Phi}\left(\frac{\xi}{\tau^2}\right). \tag{4.4}$$

One choice of mother wavelets in W_0 is then given by

$$\psi_L(x) = \frac{1}{\sqrt{2\tau}}(\phi_L(\tau^2 x) + \phi_L(\tau^2 x - 1)) - \sqrt{2}\phi_S(\tau^2 x - 2) \tag{4.5}$$

$$\psi_S(x) = \phi_L(\tau^2 x) - \tau^{1/2}\phi_S(\tau^2 x - 1) \tag{4.6}$$

or, equivalently after Fourier transform,

$$\hat{\Psi}(\xi) = \begin{pmatrix} \frac{\tau^{-5/2}}{\sqrt{2}}(1 + \mathrm{e}^{-2\pi \mathrm{i}\xi/\tau^2}) & -\sqrt{2}\tau^{-2}\mathrm{e}^{-4\pi\mathrm{i}\xi/\tau^2} \\ \tau^{-2} & -\tau^{-3/2}\mathrm{e}^{-2\pi\mathrm{i}\xi/\tau^2} \end{pmatrix} \hat{\Phi}\left(\frac{\xi}{\tau^2}\right) \equiv M_1\left(\frac{\xi}{\tau^2}\right)\hat{\Phi}\left(\frac{\xi}{\tau^2}\right). \tag{4.7}$$

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